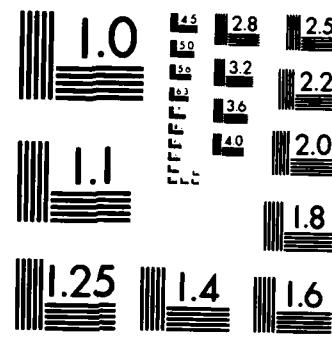


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ON LIMIT OF THE LARGEST EIGENVALUE  
OF THE LARGE DIMENSIONAL SAMPLE  
COVARIANCE MATRIX

Y. Q. Yin<sup>1</sup>, Z. D. Bai<sup>1</sup>,  
and  
P. R. Krishnaiah<sup>2</sup>

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ON LIMIT OF THE LARGEST EIGENVALUE OF THE  
LARGE DIMENSIONAL SAMPLE COVARIANCE MATRIX

Y. Q. Yin, Z. D. Bai, and P. R. Krishnaiah

ABSTRACT

In this paper, the authors showed that the largest eigenvalue of the sample covariance matrix tends to a limit under certain conditions when both the number of variables and the sample size tend to infinity. The above result is proved under the mild restriction that the fourth moment of the elements of the sample sums of squares and cross products (SP) matrix exist.

Keywords: Largest eigenvalue, sample covariance matrix, large dimensional random matrices, limit.

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## 1. INTRODUCTION

The distribution of the largest eigenvalue of the sample covariance matrix is useful in certain problems of inference in the area of multivariate analysis. For example, it is useful in testing the hypothesis that the eigenvalues of the covariance matrix are equal to a specified value. Geman (1980) showed that the largest eigenvalue of  $A = WW'/n$  tends to  $(1 + \sqrt{y})^2$  almost surely when  $\lim (p/n) = y$ ,  $W = (w_{ij})$  and  $w_{ij}$ 's are distributed independently with mean zero and variance one. In proving the above result, Geman assumed that  $E|w_{11}|^n \leq n^{\alpha n}$  for  $n = 1, 2, \dots$  and a positive constant  $\alpha$ . Jonsson (1983) announced the above result under the weaker condition that  $E(|w_{11}|^7) < \infty$  by using a "truncation" method. Recently, Silverstein (1984) proved the same result under the condition that  $E|w_{11}|^{6+\epsilon} < \infty$  where  $\epsilon > 0$  is arbitrary. In the present paper, we prove the above result under a much weaker condition that  $E|w_{11}|^4 < \infty$ .

## 2. PRELIMINARIES

The following results are needed in the sequel:

Lemma 2.1. For any  $\delta > 0$ , we have

$$\binom{k}{r} \binom{k}{l-r} \binom{2k-l}{l} \delta^{k-l} \leq (1 + \sqrt{\delta})^{2k} \binom{k}{l} \binom{2l}{2r}$$

for all  $1 \leq r \leq l \leq k$ , where  $\binom{n}{m} = n! / m!(n-m)!$ .

Proof. Let

$$I(r) = \frac{\binom{k}{r} \binom{k}{l-r} \binom{2l}{l}}{\binom{2l}{2r}} \text{ for } r = 0, 1, \dots, l.$$

Then, we have for  $r = 0, 1, 2, \dots, l-1$ ,

$$\frac{I(r+1)}{I(r)} = \frac{(2r+1)(k-r)}{[2(l-r)-1][k-(l-r)+1]}.$$

But

$$\begin{aligned} (2r+1)(k-r) &= [2(l-r)-1][k-(l-r)+1] \\ &= (2r-l+1)(2k-2l+1) \geq 0 \text{ iff } 2r \geq l-1. \end{aligned}$$

So,

$$\frac{I(r+1)}{I(r)} \geq 1 \text{ iff } 2r \geq l-1.$$

Hence  $I(r)$  has its maximum at  $I(0) = I(l)$ , and so

$$\begin{aligned} &\left( \binom{k}{r} \binom{k}{l-r} \binom{2k-l}{l} \right) / \left( \binom{k}{l} \binom{2l}{2r} \right) \delta^{k-l} \leq \binom{2k-l}{l} \delta^{k-l} \leq \sum_{l=0}^k \binom{2k-l}{l} \delta^{k-l} \\ &= \sum_{l=0}^k \binom{k+l}{l} \delta^l \leq \sum_{l=0}^{2k} \binom{2k}{l} \delta^{l/2} = (1 + \sqrt{\delta})^{2k}. \end{aligned}$$

In studying strong limit properties of random matrix, the techniques of truncation and centralization play an important role.

Lemma 2.2. (Truncation Lemma). Let  $r$  be a number belonging to the interval  $[\frac{1}{2}, 2]$  and let  $\{w_{ij}, i, j = 1, 2, \dots\}$  be a collection of iid random variables with  $Ew_{11} = 0$  and  $E|w_{11}|^{2/r} < \infty$ . For each  $n$ , define

$W_n$  to be  $p \times n$  matrix whose  $(i,j)$ th entry is  $w_{ij}$ , where  $p = p(n)$  is such that  $p/n \rightarrow y \in (0, \infty)$  as  $n \rightarrow \infty$ . Then there exists a sequence of positive numbers  $\delta = \delta_n$  such that

- (a)  $\delta \rightarrow 0$  as  $n \rightarrow \infty$ ,
- (b) The convergence rate of  $\delta$  can be slower than any preassigned rate,
- (c)  $P(W_n \neq \hat{W}_n, \text{ i.o.}) = 0$ .

where  $\hat{W}$  is the  $p \times n$  matrix whose  $(i,j)$ -th entry is  $\hat{w}_{ij} = w_{ij} I_{\{ |w_{ij}| < \delta n^r \}}$  and  $I_A$  denotes the indicator of the set  $A$ .

Proof. Since  $E|w_{11}|^{2/r} < \infty$ , we have for any  $\epsilon > 0$

$$\sum_{m=1}^{\infty} 2^{2m} P(|w_{11}| \geq \epsilon 2^{mr}) < \infty.$$

Because of the arbitrariness of  $\epsilon$  in the above inequality, there exists a sequence of positive numbers  $\epsilon = \epsilon_m$  such that

- (1)  $\epsilon_m \rightarrow 0$ , when  $m \rightarrow \infty$ ,
- (2) The rate of convergence is slower than any preassigned rate,
- (3)  $\sum_{m=1}^{\infty} 2^{2m} P(|w_{11}| \geq \epsilon_m 2^{mr}) < \infty$ .

Define  $\delta = \delta_n = 2\epsilon_m$  for  $2^{m-1} \leq n < 2^m$ . It is obvious that such a sequence of  $\delta$  satisfies the requirements (a) and (b). Define  $\hat{W}_n$  with this  $\delta$ . Because  $\frac{p}{n} \rightarrow y$ ,  $0 < y < \infty$ , we have  $p \leq 2yn$  when  $n$  is large enough. Thus

$$\begin{aligned} & P(W_n \neq \hat{W}_n, \text{ i.o.}) \\ & \leq \lim_{k \rightarrow \infty} \sum_{m=k}^{\infty} P\left(\bigcup_{2^{m-1} \leq n < 2^m} \bigcup_{i=1}^p \bigcup_{j=1}^n (|w_{ij}| \geq \delta n^r)\right) \\ & \leq \lim_{k \rightarrow \infty} \sum_{m=k}^{\infty} P\left(\bigcup_{2^{m-1} \leq n < 2^m} 2y 2^m \bigcup_{i=1}^p \bigcup_{j=1}^n (|w_{ij}| \geq \epsilon_m 2^{mr})\right) \\ & = \lim_{k \rightarrow \infty} \sum_{m=k}^{\infty} P\left(\bigcup_{i=1}^p \bigcup_{j=1}^n (|w_{ij}| \geq \epsilon_m 2^{mr})\right) \\ & \leq \lim_{k \rightarrow \infty} 2y \sum_{m=k}^{\infty} 2^{2m} P(|w_{11}| \geq \epsilon_m 2^{mr}) = 0 \end{aligned}$$

which completes the proof.

Lemma 2.3. (Centralization Lemma) Under the assumptions of Lemma 2.2, we have

$$|\tilde{\lambda}_{\max}(n) - \hat{\lambda}_{\max}(n)| \rightarrow 0, \text{ a.s.} \quad (2.1)$$

where  $\tilde{\lambda}_{\max}(n)$  and  $\hat{\lambda}_{\max}(n)$  are the largest eigenvalues of  $\frac{1}{2r} \tilde{W}_n \tilde{W}_n'$  and  $\frac{1}{n} \hat{W}_n \hat{W}_n'$ , respectively, and  $\tilde{W}_n$  is the  $n \times n$  matrix whose  $(i,j)$ th entry is  $\tilde{w}_{ijn} = E\hat{w}_{ijn}$ .

Proof. Denote by  $M_n$  the  $n \times n$  matrix whose entries are all  $E\hat{w}_{11n}$ .

Since

$$|E\hat{w}_{11n}| \leq E|w_{11}| I[|w_{11}| > \delta\sqrt{n}] \leq E|w_{11}|^{2/r} (\delta\sqrt{n})^{-\frac{2}{r}+1},$$

we obtain

$$\begin{aligned} & |\tilde{\lambda}_{\max}(n) - \hat{\lambda}_{\max}(n)| \\ & \leq n^{-2r} \left\{ 2 \sup_{\|a\|=1} |a' \hat{W}_n M_n a| + \sup_{\|a\|=1} |a' M_n^2 a| \right\} \\ & = n^{-2r} \left\{ 2 \sup_{\substack{\sum a_i^2=1 \\ i=1}} \left| \sum_{i=1}^p a_i \left| \sum_{i=1}^p \sum_{j=1}^n a_i \hat{w}_{ijn} E\hat{w}_{11n} \right| + n |E\hat{w}_{11n}|^2 \right. \right. \\ & \quad \times \sup_{\substack{\sum a_i^2=1 \\ i=1}} \left( \sum_{i=1}^p a_i \right)^2 \left. \right\} \\ & \leq cn^{-2r} \left\{ n^{-\frac{1}{r}+1} \delta^{-\frac{2}{r}+1} \left( \sum_{i=1}^p \sum_{j=1}^n w_{ij}^2 \right)^{1/2} + n^{-\frac{2}{r}+3} \delta^{-\frac{4}{r}+2} \right\} \\ & \leq c \left\{ n^{-\frac{1}{r}(r-1)^2-1} \delta^{-\frac{2}{r}+1} \left( n^{-2r} \sum_{i=1}^p \sum_{j=1}^n w_{ij}^2 \right)^{1/2} \right. \\ & \quad \left. + n^{-\frac{2}{r}(r-1)^2-1} \delta^{-\frac{4}{r}+2} \right\} \rightarrow 0, \text{ a.s. if } r > 1. \end{aligned}$$

$$\leq c \left\{ n^{-\frac{1}{r}(r-1)^2-r} \delta^{-\frac{2}{r}+1} \left( n^{-2} \sum_{i=1}^p \sum_{j=1}^n w_{ij}^2 \right)^{1/2} \right. \\ \left. + n^{-\frac{2}{r}(r-1)^2-1} \delta^{-\frac{4}{r}+2} \right\} \rightarrow 0, \text{ a.s. if } r \leq 1. \quad (2.2)$$

In proving the above result, we have used the facts that

$$n^{-2r} \sum_{i=1}^p \sum_{j=1}^n w_{ij}^2 \rightarrow 0, \text{ a.s. if } r > 1,$$

and

$$n^{-2} \sum_{i=1}^p \sum_{j=1}^n w_{ij}^2 \rightarrow E w_n^2 \text{ a.s. if } r \leq 1,$$

which can be seen from Marcinkiewicz strong law of large numbers.

Remark 1. Throughout this paper, we will use Lemmas 2.2 and 2.3 with  $r = \frac{1}{2}$ , and the requirement (b) is specified as

$$(b') \delta \log n \rightarrow \infty, \text{ when } n \rightarrow \infty. \quad (2.3)$$

Remark 2. From Lemmas 2.2 and 2.3 we can easily see that

$$\lambda_{\max}(n) - \tilde{\lambda}_{\max}(n) \rightarrow 0, \text{ a.s. as } n \rightarrow \infty,$$

if the conditions of Theorem 1 hold, where  $\lambda_{\max}(n)$  is the largest eigenvalue of  $\frac{1}{n} W_n W_n'$

## 3. SOME RESULTS ON GRAPH THEORY

Given a sequence  $(i_1, j_1, i_2, j_2, \dots, i_k, j_k)$ , where  $i_1, \dots, i_k$  are integers in the set  $\{1, \dots, p\}$ , and  $j_1, \dots, j_k$  are integers in the set  $\{1, \dots, n\}$ , we define a directed multigraph as follows. We draw two parallel real lines, I-line and J-line. We plot  $i_1, \dots, i_k$  on the I-line and plot  $j_1, \dots, j_k$  on the J-line. These are vertices and they are split into two disjoint classes on the two parallel lines. So even if the two integers  $i_a$  and  $j_b$  are equal, they will not be the same vertex because  $i_a$  belongs to I-line and  $j_b$  belongs to J-line. But if  $i_a = i_b$  (or  $j_a = j_b$ ), we regard these two vertices identical. Edges of the directed bigraph will be the directed segments  $\overrightarrow{i_1 j_1}$ ,  $\overrightarrow{j_1 i_2}$ ,  $\overrightarrow{i_2 j_2}$ ,  $\overrightarrow{j_2 i_3}$ ,  $\dots$ ,  $\overrightarrow{i_k j_k}$ ,  $\overrightarrow{j_k i_1}$ . They are  $2k$  in number and they should be regarded different from each other, even in the case when two edges have the same initials and ends.

Sometimes it is convenient to denote  $i_a$  by  $v_{2a-1}$  and  $j_a$  by  $v_{2a}$ . So the vertices of the graph are  $v_1, v_2, \dots, v_{2k}$ , and the edges are  $v_1 v_2, v_2 v_3, \dots, v_{2k-1} v_{2k}, v_{2k} v_1$ . Notice that when we write an edge as  $v_a v_{a+1}$ , we always mean that  $v_a$  is the initial vertex and  $v_{a+1}$  is the end vertex, the direction of the edge is from  $v_a$  to  $v_{a+1}$ .

When two edges  $v_a v_{a+1}, v_b v_{b+1}$  have the same vertex sets, i.e.  $\{v_a, v_{a+1}\} = \{v_b, v_{b+1}\}$ , we cannot conclude that  $v_a v_{a+1} = v_b v_{b+1}$ , since  $v_a v_{a+1} = v_b v_{b+1} \Leftrightarrow a = b$ . When  $\{v_a, v_{a+1}\} = \{v_b, v_{b+1}\}$ , we say that the two edges coincide.

The graph we just constructed will be called a W-graph.

A W-graph will be called canonical, if  $v_a \leq \max\{v_{a-2}, v_{a-4}, \dots\} + 1$ , for each  $a > 2$ , and  $v_1 = 1, v_2 = 1$ .

In the following, we will get a bound for the number of canonical W-graphs.

In a canonical W-graph, an edge  $v_{a-1}v_a$  ( $a \geq 2$ ) will be called an innovation, if  $v_a$  does not occur in  $v_1, v_2, \dots, v_{a-1}$ . Suppose  $v_{a-1}v_a$  is an innovation. If  $a$  is odd,  $v_{a-1}v_a$  is called a row innovation, and if  $a$  is even, a column innovation. Note that  $v_1v_2$  is a column innovation according to the above definition.

In a W-graph, an edge  $v_{a-1}v_a$  ( $a \geq 2$ ) is said to be single up to  $v_b$  ( $b \geq a$ ), if there is no edge  $v_{c-1}v_c$  with  $1 < c \leq b$  such that  $v_{c-1}v_c$  coincides with  $v_{a-1}v_a$ . An edge  $v_{b-1}v_b$  ( $b \geq 3$ ) will be called a  $T_3$ -edge if there is an innovation  $v_{a-1}v_a$ , single up to  $v_{b-1}$ , and  $v_{b-1}v_b$ ,  $v_{a-1}v_a$  coincide.

An edge will be called a  $T_4$ -edge if it is not an innovation and not a  $T_3$ -edge.

A consecutive segment  $v_a v_{a+1} \dots v_{b-1} v_b$  of the whole W-graph will be called a chain.

Lemma 3.1. Let  $v_a v_{a+1} \dots v_c$  be a chain such that

- (1)  $v_a v_{a+1}$  is single up to  $v_c$ ,
- (2)  $v_c$  has been visited by  $v_1 v_2 \dots v_a$ .

Then the chain contains at least one  $T_4$ -edge.

Proof. When  $c - a = 1$ , evidently  $v_a v_{a+1} = v_a v_c$  is an edge of  $T_4$ .

We know that  $v_{c-1}v_c$  must be a  $T_4$  or a  $T_3$  since it cannot be an innovation. If it is a  $T_4$ , the proof is completed. If  $v_{c-1}v_c$  is a  $T_3$ , then there is a single innovation  $v_{b-1}v_b$  coincident with  $v_{c-1}v_c$  such that  $b < c$ .

Case 1.  $b \geq a+1$ . Since  $v_c = v_{b-1}$  or  $v_c = v_b$ , then either  $v_a v_{a+1} \dots v_{b-1}$  or  $v_a v_{a+1} \dots v_b$  has the properties (1), (2) and is shorter than  $v_a v_{a+1} \dots v_c$ . By induction hypothesis  $v_a v_{a+1} \dots v_b$  contains a  $T_4$ -edge, but it is a part of the chain  $v_a v_{a+1} \dots v_c$ . So the original path contains a  $T_4$ -edge.

Case 2.  $b < a+1$ , i.e.  $v_{b-1}v_b$  is in the path  $v_1v_2\dots v_a$ , and then  $v_{c-1} = (v_b \text{ or } v_{b-1})$  is visited by  $v_1v_2\dots v_a$ . Thus  $v_a v_{a+1}\dots v_{c-1}$  has the properties (1),(2). By induction, the lemma is proved.

Note: In a W-graph, the chain  $v_1v_2\dots v_{a+1}$  determines completely whether the edge  $v_a v_{a+1}$  is an innovation, a  $T_3$  or a  $T_4$ .

Lemma 3.2. If in the chain  $v_1v_2\dots v_a$ , there are  $s$  edges, each of which is single up to  $v_a$  and has a vertex equal to  $v_a$  and if  $t$  is the number of noncoincident  $T_4$  edges in  $v_1v_2\dots v_a$ , then  $s \leq t + 1$ .

Proof. Let  $v_{a_1-1}v_{a_1}, v_{a_2}v_{a_2+1}, \dots, v_{a_s}v_{a_s+1}$  be all the single edges such that  $a_1 < a_2 < \dots < a_s < a$  and  $v_{a_2}v_{a_2+1}, \dots, v_{a_s}v_{a_s+1}$  are single up to  $v_a$ , and

$$v_{a_1} = v_{a_2} = \dots = v_{a_s} = v_a.$$

Consider chains  $L_2 = v_{a_2}v_{a_2+1} \dots v_{a_3}$ ,  $L_3 = v_{a_3}v_{a_3+1} \dots v_{a_4}$ , ...,  $L_s = v_{a_s}v_{a_s+1} \dots v_a$ . By Lemma 3.1,  $L_2$  has a  $T_4$ -edge  $E_2$ . Let  $v_{b_3}$  be the first vertex in  $L_3$  which belongs to  $L_2$ . Then by Lemma 3.1

$v_{a_3}v_{a_3+1} \dots v_{b_3}$  contains an edge of  $T_4$  and we denote it by  $E_3$ .

Evidently,  $E_3$  and  $E_2$  are not coincident. Let  $v_{b_4}$  be the first vertex

of  $L_4$  which also belongs to  $L_2 \cup L_3$ . Then  $v_{a_4}v_{a_4+1} \dots v_{b_4}$  has an edge of  $T_4$ , by Lemma 3.1. Let it be denoted by  $E_4$ . Evidently no two of

$E_2, E_3, E_4$  are coincident. Continue this procedure. Finally, we get

$s-1$  edges of  $T_4$ , which are not coincident with each other. So

$$s-1 \leq t, s \leq t+1.$$

A  $T_3$  edge  $v_a v_{a+1}$  is called regular, if there is more than one innovation with a vertex equal to  $v_a$  and single up to  $v_a$ .

Lemma 3.3. In any W-graph there is a mapping  $\Phi$  from regular  $T_3$  edges to  $T_4$  edges such that for any  $T_4$  edge  $E$ , there are at most two regular  $T_3$  edges whose  $\Phi$  image is  $E$ .

Proof. Define  $\Phi$  as follows.

Let  $v_{a_1} v_{a_1+1}, v_{a_2} v_{a_2+1}, \dots, v_{a_s} v_{a_s+1}$  be the set of all innovations single up to  $v_a$  such that

$$(1) \quad v_{a_2} = v_{a_3} = \dots = v_{a_s} = v_a$$

$$(2) \quad v_{a_1} = v_a \text{ or } v_{a_1+1} = v_a$$

$$(3) \quad a_1 < a_2 < \dots < a_s < a_{s+1} = a.$$

We note that there is at most one innovation inward to  $v_a$  and if there is such an innovation, it must be the foremost one among innovations with a vertex  $v_a$ .

In this W-graph,  $v_a v_{a+1}$  must coincide with one of  $v_{a_1} v_{a_1+1}, \dots, v_{a_s} v_{a_s+1}$ . Let that one be  $v_{a^*} v_{a^*+1}$ . Also, let

$$v = v(a) = \begin{cases} i+1, & \text{if } a^* = a_i, \text{ for } i = 1, \dots, s-1, \\ s, & \text{if } a^* = a_s. \end{cases}$$

By Lemma 3.1, in the path  $C_a = v_{a_1} v_{a_1+1} \dots v_{a_s+1}$ , there is at least one edge of  $T_4$ . Let the first one be  $E_a$ . Define  $\Phi(v_a v_{a+1}) = E_a$ .

At first we prove that if  $v_a v_{a+1}, v_b v_{b+1}$  are two regular  $T_3$  edges and  $v_a \neq v_b$  then  $E_a \neq E_b$  ( $E_a, E_b$  may coincide). Suppose  $C_a = v_{a_1} v_{a_1+1} \dots v_{a_s+1}$ ,

$C_b = v_{b_1} v_{b_1+1} \dots v_{b_s+1}$ . Then, we have the following possibilities to consider:

$$(1) \quad a' < a'' < b' < b'',$$

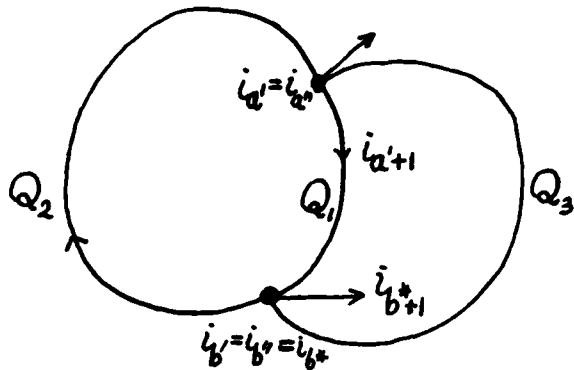
$$(2) \quad a' < b' < a'' < b'',$$

$$(3) \quad a' < b' < b'' < a''.$$

For case (1),  $E_a \neq E_b$  is evident.

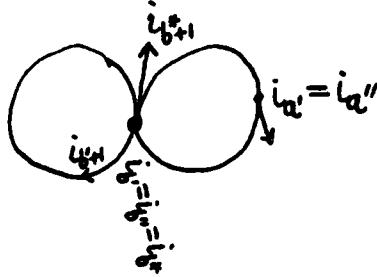
Consider case (2).  $C_a, C_b$  are divided into three parts as

$Q_1 = v_{a_1} v_{a_1+1} \dots v_{a_s+1}$ ,  $Q_2 = v_{b_1} v_{b_1+1} \dots v_{b_s+1}$ ,  $Q_3 = v_{a_s} v_{a_s+1} \dots v_{b_s+1}$  as given in the following diagram.



It is enough to show that  $E_a$  is in  $Q_1$ ,  $E_b$  is in  $Q_2 \cup Q_3$ . By definition, there is an innovation  $v_{b*}v_{b*+1}$  with  $b^* < b'$ ,  $v_{b*} = v_{b'} = v_b$ , single up to  $v_b$ . If  $b^* \leq a'$ , by Lemma 3.1,  $Q_1$  contains a  $T_4$ -edge. If  $a' < b^* < b'$ , we can consider the chain  $v_{b*}v_{b*+1} \dots v_{b'}$ ; it is part of  $Q_1$  and it contains a  $T_4$ -edge by Lemma 3.1. So  $E_a \in Q_1$ . It is obvious that  $E_b$  is in  $Q_2 \cup Q_3$ . Thus  $E_a \neq E_b$ .

Consider (3). As before, we can show that  $E_a$  is in  $v_{a*}v_{a'+1} \dots v_b$  and  $E_b$  is in  $v_{b*}v_{b'+1} \dots v_{b''}$ .



Then we consider the case  $v_a = v_b$  and  $a < b$ . Now  $v_a v_{a+1}$ ,  $v_b v_{b+1}$  both are regular  $T_3$  edges and so they coincide with single innovations  $v_{a*}v_{a*+1}$  and  $v_{b*}v_{b*+1}$ .  $v_{a*}v_{a*+1}$  and  $v_{b*}v_{b*+1}$  cannot coincide. So  $E_a \neq E_b$  except  $v_{a*}v_{a*+1}$ ,  $v_{b*}v_{b*+1}$  are the last two single innovations. In the last case  $\Phi^{-1}(E_a)$  has cardinal 2.

At last we get that the mapping  $\Phi: v_a v_{a+1} \longrightarrow E_a$  for regular  $T_3$ 's has the property that  $\Phi^{-1}(E_a)$  has at most two edges of  $T_3$ . The proof of Lemma 3.3 is completed.

## 4. LIMIT OF THE LARGEST EIGENVALUE

We now prove the main result of our paper.

Theorem 3.1. Let  $\{w_{ij} : i, j = 1, 2, \dots\}$  be an infinite matrix of iid random variables,  $Ew_{11} = 0$  and  $Ew_{11}^4 < \infty$ . If  $\lambda_{\max}(n)$  denotes the largest eigenvalue of the matrix  $\frac{1}{n} W_n W_n'$ , here  $W_n$  denotes the  $p \times n$  random matrix  $[w_{ij} : i = 1, \dots, p; j = 1, \dots, n]$ , then

$$\lim_{n \rightarrow \infty} \lambda_{\max}(n) = (1 + \sqrt{y})^2 Ew_{11}^2 \text{ a.s.}$$

as  $n \rightarrow \infty$ ,  $p \rightarrow \infty$  and  $p/n \rightarrow y$ .

Proof. Without loss of generality, we assume that  $Ew_{11}^2 = 1$ . We only have to prove that  $\overline{\lim} \lambda_{\max}(n) \leq (1 + \sqrt{y})^2$  a.s. . But, according to Remark 2 of Section 2, it is sufficient to show that

$$\overline{\lim} \tilde{\lambda}_{\max}(n) \leq (1 + \sqrt{y})^2 \text{ a.s. .}$$

In other words, we assume in the sequel that

$$(1) \quad |w_{ij}| < \delta \sqrt{n},$$

$$(2) \quad Ew_{ij} = 0,$$

$$(3) \quad Ew_{ij}^2 \leq 1,$$

$$(4) \quad E|w_{ij}|^l \leq (\delta \sqrt{n})^{l-1}, \text{ for } l \geq 2,$$

$$(5) \quad E|w_{ij}|^l \leq c(\delta \sqrt{n})^{l-3}, \text{ for } l \geq 3.$$

Now, choose  $z > (1 + \sqrt{y})^2$  arbitrarily. We will now show that

$$(6) \quad \sum_{n=1}^{\infty} E \left( \frac{\lambda_{\max}(n)}{z} \right)^k < \infty$$

where  $k = k_n$  satisfies

$$(7) \quad k_n / \log n \longrightarrow \infty,$$

$$(8) \quad \delta^{1/6} k_n / \log n \longrightarrow 0.$$

We have

$$\begin{aligned} E[\lambda_{\max}(n)]^k &\leq E \operatorname{tr}\left(\frac{1}{n} W W^T\right)^k = n^{-k} E \operatorname{tr}(W W^T)^k \\ &= n^{-k} \sum E w_{i_1 j_1} w_{i_2 j_1} w_{i_2 j_2} \cdots w_{i_k j_k} w_{i_1 j_k}. \end{aligned}$$

Here the summation is taken in such a way that  $i_1, \dots, i_k$  run over all integers in  $\{1, \dots, p\}$ , and  $j_1, \dots, j_k$  run over all integers in  $\{1, \dots, n\}$ .

The above sum can be split in the following form.

$$E \operatorname{tr}\left(\frac{1}{n} W W^T\right)^k = n^{-k} \sum' \sum'' \sum''' E w_{i_1 j_1} w_{i_2 j_1} w_{i_2 j_2} \cdots w_{i_k j_k} w_{i_1 j_k},$$

here

$\sum'$  - summation for different arrangements of four different types of elements at the  $2k$  different positions.

$\sum''$  - summation for different canonical graphs  $\Gamma$  with given arrangement of the four types for  $2k$  positions.

$\sum'''$  - summation of  $E w_{i_1 j_1} w_{i_2 j_1} \cdots w_{i_1 j_k}$  for which the graph is isomorphic to the given canonical graph.

Let  $r$  denote the number of row innovations,  $\ell$  denote the number of  $T_3$  edges. Then there are  $\ell - r$  column innovations and  $(2k - 2\ell)T_4$  edges and so  $\sum'$  is bounded by  $\sum_{\ell=1}^k \sum_{r=1}^{\ell} \binom{k}{r} \binom{k}{\ell-r} \binom{2k-\ell}{\ell}$ . Since every row innovation leads to a free  $i$ -index and every column innovation leads to a free  $j$ -index except the first column innovation  $v_1 v_2$  which leads to an  $i$ -index and a  $j$  one. We know that  $\sum'''$  is bounded by  $p^{r+1} n^{\ell-r}$ .

To bound  $\sum''$ , let  $t$  denote the number of noncoincident  $T_4$  edges. By definition, each innovation in a canonical  $W$ -graph is uniquely determined by the chain before it and each nonregular  $T_3$  edge is so

done as innovations. If  $t = 0$ , i.e.  $\ell = k$ , then by Lemma 3.3, there are no  $T_4$  edge and regular  $T_3$  edge, so that  $\sum$  is only one summand. For  $t \geq 1$ , since each  $T_4$  edge in a canonical W-graph must be one of the  $k^2$  elements in the  $k \times k$  matrix  $(w_{ij})$   $i \leq k, j \leq k$ , all the possibilities that  $2k - 2\ell$   $T_4$  edges may take are less than  $\binom{k^2}{t} t^{2k-2\ell}$ . By Lemmas 3.2 and 3.3, all the possibilities that all the regular  $T_3$  edges take is not more than  $(t+1)^{4k-4\ell}$ . Hence  $\sum$  is bounded by  $\binom{k^2}{t} t^{2k-2\ell} (t+1)^{4k-4\ell} \leq k^{2t} (t+1)^{6k-6\ell}$ .

Finally, we bound the expectation  $E w_{i_1 j_1} w_{i_2 j_1} \dots w_{i_k j_k} w_{i_1 j_k}$ . If  $t = 0$ , each expectation is  $(E w_{11}^2)^\ell \leq 1$ . For  $t \geq 1$ , let  $\mu$  denote the number of innovations which coincide with at least one  $T_4$  edge and let  $n_i$  denote the number of  $T_4$  edges which coincide with the  $i$ -th such innovation,  $i = 1, 2, \dots, \mu$ , respectively.

Let  $m_j$  be the number of  $T_4$  edges which coincide with each other but not with any innovation,  $j = 1, 2, \dots, t-\mu$ . Then we have

$$E w_{i_1 j_1} w_{i_2 j_1} \dots w_{i_k j_k} w_{i_1 j_k} = (E w_{11}^2)^{\ell-\mu} \prod_{i=1}^{\mu} (E w_{11}^{n_i+2}) \prod_{j=1}^{t-\mu} E w_{11}^{m_j}$$

where  $2(\ell-\mu) + \sum_{i=1}^{\mu} (n_i + 2) + \sum_{j=1}^{t-\mu} m_j = 2k$  and  $\mu \leq t$ . By (3), (4) and

(5) we have

$$|E w_{i_1 j_1} w_{i_2 j_1} \dots w_{i_k j_k} w_{i_1 j_k}| \leq c^\mu (\delta \sqrt{n})^{\sum_{i=1}^{\mu} (n_i+1) + \sum_{j=1}^{t-\mu} (m_j+1)}$$

$$= c^\mu (\delta \sqrt{n})^{2k-2\ell-t} \leq k^t (\delta \sqrt{n})^{2k-2\ell-t}$$

when  $k$  is large enough.

By the above argument, we obtain

$$E \operatorname{tr} \left( \frac{1}{n} W_n W_n^T \right)^k \leq n^{-k} \sum_{\ell=1}^k \sum_{r=1}^{\ell} \binom{k}{r} \binom{k}{\ell-r} \binom{2k-\ell}{\ell} p^{r+1} n^{\ell-r} \left[ \sum_{t=0}^{2k-2\ell} k^{3t} (t+1)^{6k-6\ell} \right] \delta^{2k-2\ell-t}$$

$$\leq p^k \sum_{\ell=1}^k \sum_{r=1}^{\ell} \binom{k}{r} \binom{k}{\ell-r} \binom{2k-\ell}{\ell} \left( \frac{p}{n} \right)^r \left[ \sum_{t=0}^{2k-2\ell} k^{3t} (t+1)^{6k-6\ell} (\delta \sqrt{n})^{-t} \right] \delta^{2(k-\ell)}$$

Using the elementary inequality  $a^{-t} (t+1)^b \leq \left( \frac{b}{\log a} \right)^b$  for  $a > 1, b > 0$ ,  
 $t > 0$ , we have

$$\begin{aligned} \sum_{t=0}^{2k-2\ell} k^{3t} (t+1)^{6k-6\ell} (\delta \sqrt{n})^{-t} &\leq 2k \left( \frac{6k-6\ell}{\log \frac{\delta \sqrt{n}}{k^3}} \right)^{6k-6\ell} \\ &\leq 2k \left( \frac{6k}{\frac{1}{2} \log n + \log \delta - 3 \log k} \right)^{6(k-\ell)} \\ &\leq 2k \left( \frac{18k}{\log n} \right)^{6(k-\ell)} \quad \text{when } n \text{ is large enough.} \end{aligned}$$

Using Lemma 2.1, we get for all large  $n$

$$\begin{aligned} E \operatorname{tr} \left( \frac{1}{n} W_n W_n^T \right)^k &\leq 2kp \sum_{\ell=1}^k \sum_{r=1}^{\ell} (1+\sqrt{\delta})^{2k} \binom{k}{\ell} \binom{2\ell}{2r} \left( \frac{p}{n} \right)^r \left[ \frac{18\delta^{1/6} k}{\log n} \right]^{6(k-\ell)} \\ &\leq 2kp (1+\sqrt{\delta})^{2k} \left[ (1+\sqrt{p/n})^2 + \left[ \frac{18\delta^{1/6} k}{\log n} \right]^6 \right]^k \\ &= (2kp)^{1/k} (1+\sqrt{\delta})^2 \left[ (1+\sqrt{p/n})^2 + \left[ \frac{18\delta^{1/6} k}{\log n} \right]^6 \right]^k \\ &\leq n^k, \end{aligned}$$

where  $n$  is a constant satisfying  $(1+\sqrt{y})^2 < n < z$ . Here the last inequality follows from the following facts:

- (a)  $(2kp)^{1/k} \rightarrow 1$ , because  $k/\log n \rightarrow \infty$  and  $p/n \rightarrow y \in (0, \infty)$
- (b)  $(1+\sqrt{\delta})^2 \rightarrow 1$ , because  $\delta \rightarrow 0$
- (c)  $(1+\sqrt{p/n})^2 \rightarrow (1+\sqrt{y})^2$ , because  $p/n \rightarrow y \in (0, \infty)$
- (d)  $\left[ \frac{18\delta^{1/6} k}{\log n} \right]^6 \rightarrow 0$ , because  $\delta^{1/6} k / \log n \rightarrow 0$ .

This leads to (6) since  $k/\log n \rightarrow \infty$  and the proof is thus complete.

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